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On the Green function for a hydrogen-like atom in the Dirac monopole field plus the Aharonov–Bohm field

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Abstract. The connection between the Green function for an isotropic harmonic oscillator in two-dimensional complex space and that for a hydrogen-like atom in the Dirac monopole field plus the Aharonov–Bohm field is established by using path integrals. The connection found provides a simple method for constructing the Green function for atomic systems.

1. Introduction

The method of functional integration, intensively developed by Feynman for the problems of quantum mechanics and quantum field theory, is used in the majority of areas of theoretical physics, see, for example, Slavnov and Fadeev (1978). However, the number of physical systems which allow an exact solution by the use of this method is very limited, therefore the exact calculation of path integrals for any non-Gaussian system is always of considerable interest. This is evident from lots of recent works devoted to exact path integrals of quantum systems; see, for example, Duru and Kleinert (1979), Ho Roger and Inomata (1982), Sökmen (1986), Florencio and Goodman (1986), Chetouani *et al* (1989). Ho Roger and Inomata (1982) have established exact path integrals for a hydrogen-like atom by using the connection between the hydrogen-like atom problem and the problem of a four-dimensional harmonic oscillator (Bergmann and Frishman 1965). By this treatment the simple representation of the Coulomb Green function becomes very suitable to use in concrete calculations (Le Van Hoang *et al* 1989). The exact path integrals for a dyon (the problem of a charged particle moving in the Coulomb field plus the Dirac monopole field) have been calculated by Chetouani *et al* (1990). By using path integrals, Le Van Hoang (1992) established the Green function for the Mic–Kepler problem (the particle in the Coulomb field plus the Dirac monopole field plus the scalar potential $-\mu/r^2$) from the Green function of a harmonic oscillator. In the present paper we will establish the connection between the Green function for an isotropic harmonic oscillator in two-dimensional complex space and that for a hydrogen atom in both the Dirac monopole field and the Aharonov–Bohm field. The connection found, firstly, follows a simple recipe in constructing the Green function for the above-mentioned atomic system and, secondly, allows us to use the useful representation of the Green function in concrete calculations.

2. The connection between the Green functions

Let us consider the Schrödinger equation in two-dimensional space with the complex coordinates ξ_s ($s=1,2$; we assume ξ_s to be the spinor components):

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial\xi_s\,\partial\xi_s^*}+\frac{1}{2}\,\omega^2\xi_s\xi_s^*\right)\Psi(\xi)=Z\Psi(\xi) \tag{1}$$

where the asterisk denotes the complex conjugate operation with summation over repeated indices. In (1) ω is a real positive number. The Green function for (1) in the energy representation is a solution of the following equation:

$$\left(Z+\frac{1}{2}\frac{\partial^2}{\partial\xi_s\,\partial\xi_s^*}-\frac{1}{2}\,\omega^2\xi_s\xi_s^*\right)U(\xi,\eta;Z)$$
$$=\mathrm{i}\delta(\xi_1'-\eta_1')\delta(\xi_1''-\eta_1'')\delta(\xi_2'-\eta_2')\delta(\xi_2''-\eta_2'') \tag{2}$$

where $\xi_s'=\operatorname{Re}\xi_s$, $\xi_s''=\operatorname{Im}\xi_s$; $\delta(x)$ is a Dirac delta-function. One of the ways of constructing the function $U(\xi,\eta;Z)$ is to represent it as a path integral—see, for example, Slavnov and Fadeev (1978):

$$U(\xi,\eta;Z)=\int_0^\infty \mathrm{d}\vartheta\,\mathrm{e}^{\mathrm{i}Z\vartheta}\prod_{\vartheta'}\int \mathrm{D}^4\xi(\vartheta')$$
$$\times\exp\left\{\mathrm{i}\int_0^\vartheta \mathrm{d}\vartheta'(2\dot\xi_s(\vartheta')\dot\xi_s^*(\vartheta')-\tfrac{1}{2}\,\omega^2\xi_s(\vartheta')\xi_s^*(\vartheta'))\right\} \tag{3}$$

where $\dot\xi_s(\vartheta')=\partial\xi_s(\vartheta')/\partial\vartheta'$ and $\xi_s(\vartheta)=\xi_s$, $\xi_s(0)=\eta_s$. Equation (3) is regarded as a limit (when $\varepsilon\to 0$, $N\to\infty$, $N\varepsilon=\vartheta$) of the following expression

$$U(\xi,\eta;Z)=\int_0^\infty \mathrm{d}\vartheta\lim_{\substack{\varepsilon\to 0\\ N\to\infty}}\left(\frac{2}{\mathrm{i}\pi\varepsilon}\right)^{2N}\prod_{k=1}^{N-1}\int \mathrm{d}^4\xi(k)$$
$$\times\exp\left\{\mathrm{i}\sum_{k=1}^{N}\left[\frac{2}{\varepsilon}(\xi_s^*(k)-\xi_s^*(k-1))(\xi_s(k)-\xi_s(k-1))\right.\right.$$
$$\left.\left.+Z\varepsilon-\tfrac{1}{2}\,\omega^2\varepsilon\xi_s^*(k-1)\xi_s(k-1)\right]\right\} \tag{4}$$

where $\xi_s(N)=\xi_s$, $\xi_s(0)=\eta_s$ and $\mathrm{d}^4\xi(k)=\mathrm{d}\xi_1'(k)\,\mathrm{d}\xi_1''(k)\,\mathrm{d}\xi_2'(k)\,\mathrm{d}\xi_2''(k)$; for brevity, we use the notation $\xi_s(k)\equiv\xi_s(k\varepsilon)$. In order to establish the relationship between the function $U(\xi,\eta;Z)$ and the Green function for a hydrogen-like atom in both the Dirac monopole field and the Aharonov–Bohm field we change in (4) the variables, choosing as the new variables the following functions:

$$x_\lambda(\vartheta)=(\sigma_\lambda)_{st}\xi_s^*(\vartheta)\xi_t(\vartheta)\qquad \phi(\vartheta)=\nu\arg(\xi_1)+\mu\arg(\xi_2) \tag{5}$$

where $\nu+\mu=1$; $\sigma_\lambda(\lambda=1,2,3)$ are the Pauli matrices. In view of the assumed properties of ξ_s the functions $x_\lambda(\vartheta)$ constitute the components of a three-dimensional vector function. Making use of the relation

$$(\sigma_\lambda)_{st}(\sigma_\lambda)_{uv}=2\delta_{sv}\delta_{tu}-\delta_{st}\delta_{uv} \tag{6}$$

one can easily ascertain that

$$\dot{\xi}_s(\vartheta)\dot{\xi}_s^*(\vartheta)=\frac{1}{4r(\vartheta)}\dot{x}_\lambda(\vartheta)\dot{x}_\lambda(\vartheta)-\frac{1}{4r(\vartheta)}[\dot{\xi}_s^*(\vartheta)\xi_s(\vartheta)-\xi_s^*(\vartheta)\dot{\xi}_s(\vartheta)]^2$$

$$=\frac{1}{4r(\vartheta)}\dot{x}_\lambda(\vartheta)\dot{x}_\lambda(\vartheta)-r(\vartheta)[\dot{\phi}(\vartheta)-\dot{x}_\lambda(\vartheta)A_\lambda(\boldsymbol{r}(\vartheta))]^2 \tag{7}$$

$$\dot{x}_\lambda(\vartheta)=\frac{\mathrm{d}x_\lambda(\vartheta)}{\mathrm{d}\vartheta} \qquad \dot{\phi}(\vartheta)=\frac{\mathrm{d}\phi(\vartheta)}{\mathrm{d}\vartheta}$$

$$r(\vartheta)=(x_\lambda(\vartheta)x_\lambda(\vartheta))^{-1/2}=\xi_s^*(\vartheta)\xi_s(\vartheta)) \tag{8}$$

where the components of vector function $A_\lambda(\boldsymbol{r}(\vartheta))$ have the form

$$A_1(\boldsymbol{r}(\vartheta))=\frac{x_2(\vartheta)}{2r(\vartheta)(r(\vartheta)+x_3(\vartheta))}-\mu\frac{x_2(\vartheta)}{x_1^2(\vartheta)+x_2^2(\vartheta)}$$

$$A_2(\boldsymbol{r}(\vartheta))=-\frac{x_1(\vartheta)}{2r(\vartheta)(r(\vartheta)+x_3(\vartheta))}+\mu\frac{x_1(\vartheta)}{x_1^2(\vartheta)+x_2^2(\vartheta)} \tag{9}$$

$$A_3(\boldsymbol{r}(\vartheta))=0.$$

From (7) it follows that in calculating the path integral using (4), the appropriate substitution of variables is

$$x_\lambda(k)=(\sigma_\lambda)_{st}\xi_s^*(k)\xi_t(k)$$

$$\phi(k)=\nu\arg(\xi_1(k))+\mu\arg(\xi_2(k)) \qquad \nu+\mu=1 \tag{10}$$

$$\mathrm{d}^4\xi(k)=\frac{1}{8r(k)}\mathrm{d}^3x(k)\,\mathrm{d}\phi(k).$$

Therefore, it follows that

$$U(\xi,\eta;Z)=\int_0^\infty \mathrm{d}\vartheta \lim_{\substack{\varepsilon\to 0\\ N\to\infty}}\left(\frac{2}{\mathrm{i}\pi\varepsilon}\right)^{2N}\prod_{k=1}^{N-1}\int\frac{\mathrm{d}^3x(k)\,\mathrm{d}\phi(k)}{8r(k)}$$

$$\times\exp\Bigg\{\mathrm{i}\sum_{k=1}^{N}\Bigg[\frac{1}{2\varepsilon r(k)}(x_\lambda(k)-x_\lambda(k-1))(x_\lambda(k)-x_\lambda(k-1))$$

$$+\frac{2r(\vartheta)}{\varepsilon}\Big[\phi(k)-\phi(k-1)-(x_\lambda(k)-x_\lambda(k-1))A_\lambda(\boldsymbol{r}(k))\Big]^2$$

$$+Z\varepsilon-\tfrac{1}{2}\omega^2\varepsilon r(k-1)\Bigg]\Bigg\} \tag{11}$$

where $\boldsymbol{r}(N)=\boldsymbol{r}$, $\boldsymbol{r}(0)=\boldsymbol{r}'$, $\phi(N)=\phi$, $\phi(0)=\phi'$. Given that $U(\xi,\eta;Z)$ is the integral operator which will be used in the class of functions depending on ϕ as follows

$$\Psi(\boldsymbol{r},\phi)=\psi(\boldsymbol{r})\,\mathrm{e}^{2\mathrm{i}q\phi} \qquad q=0,\pm\tfrac{1}{2},\pm 1,\ldots \tag{12}$$

(Le Van Hoang and Viloria 1992) we consider the following function

$$K_q(\boldsymbol{r}, \boldsymbol{r}'; -\tfrac{1}{2}\omega^2)$$
$$=\frac{1}{8}\int_0^{2\pi} \mathrm{d}\phi\, U(\xi, \eta; Z)\, \mathrm{e}^{2\mathrm{i}q(\phi-\phi')}$$
$$=\frac{1}{8}\int_0^{\infty} \mathrm{d}\vartheta \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \left(\frac{2}{\mathrm{i}\pi\varepsilon}\right)^{2N} \int_0^{2\pi} \mathrm{d}\phi(N) \prod_{k=1}^{N-1} \int \frac{\mathrm{d}^3x(k)\, \mathrm{d}\phi(k)}{8r(k)}$$
$$\times \exp\left\{\mathrm{i} \sum_{k=1}^{N} \left[\frac{1}{2\varepsilon r(k)}(x_\lambda(k)-x_\lambda(k-1))(x_\lambda(k)-x_\lambda(k-1))\right.\right.$$
$$+\frac{2r(\vartheta)}{\varepsilon}[\phi(k)-\phi(k-1)-(x_\lambda(k)-x_\lambda(k-1))A_\lambda(\boldsymbol{r}(k))]^2$$
$$\left.\left.+2q(\phi(k)-\phi(k-1))+Z\varepsilon-\tfrac{1}{2}\omega^2\varepsilon r(k-1)\right]\right\}. \tag{13}$$

Integrating over the variables $\phi(k)$ $(k=1,2,\ldots,N)$, we find

$$K_q(\boldsymbol{r}, \boldsymbol{r}'; -\tfrac{1}{2}\omega^2)$$
$$=\int_0^{\infty} \mathrm{d}\vartheta\, r(N) \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \frac{1}{(2\mathrm{i}\pi\varepsilon r(N))^{\frac{3}{2}}} \prod_{k=1}^{N-1} \int \frac{\mathrm{d}^3x(k)}{(2\mathrm{i}\pi\varepsilon r(k))^{\frac{3}{2}}}$$
$$\times \exp\left\{\mathrm{i} \sum_{k=1}^{N} \left[\frac{1}{2\varepsilon r(k)}(x_\lambda(k)-x_\lambda(k-1))(x_\lambda(k)-x_\lambda(k-1))\right.\right.$$
$$\left.\left.+2q(x_\lambda(k)-x_\lambda(k-1))A_\lambda(\boldsymbol{r}(k))+\frac{q^2\varepsilon}{2r(k)}+Z\varepsilon-\tfrac{1}{2}\omega^2\varepsilon r(k-1)\right]\right\} \tag{14}$$

where

$$\boldsymbol{r}=\xi_s^*(\boldsymbol{\sigma})_{st}\xi_t \qquad \boldsymbol{r}'=\eta_s^*(\boldsymbol{\sigma})_{st}\eta_t. \tag{15}$$

The weight $\frac{1}{8}$ in the integration over $\phi(N)$ is defined by the Jacobian of transformation (10) and the condition of normalization of wavefunctions (Le Van Hoang 1992a).

The last step is the change of 'time' variable. Let

$$\varepsilon(k)=\varepsilon r(k) \tag{16}$$

which in the limit $\varepsilon\to 0$ is equivalent to the introduction of a new time variable

$$t=\int_0^{\vartheta} \mathrm{d}\vartheta'\, r(\vartheta'). \tag{17}$$

As a result we have

$$K_q(\boldsymbol{r}, \boldsymbol{r}'; -\tfrac{1}{2}\omega^2)$$
$$=\int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\frac{1}{2}\omega^2 t} \prod_{\tau} \int \mathrm{d}^3x(\tau)$$
$$\times \exp\left\{\mathrm{i}\int_0^t \mathrm{d}\tau\left(\frac{1}{2}\dot{x}_\lambda(\tau)\dot{x}_\lambda(\tau)+\frac{Z}{r(\tau)}+2q\dot{x}_\lambda(\tau)A_\lambda(\boldsymbol{r}(\tau))-\frac{q^2}{2r^2(t)}\right)\right\} \tag{18}$$

where $\boldsymbol{r}(t)=\boldsymbol{r}$, $\boldsymbol{r}(0)=\boldsymbol{r}'$. Denoting $E=-\frac{1}{2}\omega^2$, we obtain from (18)

$$\left\{E-\frac{1}{2}(\boldsymbol{p}+\boldsymbol{A})^2+\frac{Z}{r}-\frac{q^2}{2r^2}\right\}K_q(\boldsymbol{r}, \boldsymbol{r}'; E)=\mathrm{i}\delta(\boldsymbol{r}-\boldsymbol{r}') \tag{19}$$

where $\boldsymbol{p}$ is an impulse operator; $\boldsymbol{A}$ is a vector potential formed by both the Dirac monopole potential and the Aharonov–Bohm potential is as follows:

$$A_1(\boldsymbol{r}) = q\frac{x_2}{2r(r+x_3)} - 2\mu q\frac{x_2}{x_1^2+x_2^2}$$
$$A_2(\boldsymbol{r}) = -q\frac{x_1}{2r(r+x_3)} + 2\mu q\frac{x_1}{x_1^2+x_2^2} \tag{20}$$
$$A_3(\boldsymbol{r}) = 0.$$

Here the term $2\mu q$ is a magnetic flux inside the solenoid. From (19) it follows that the function

$$K_q(\boldsymbol{r}, \boldsymbol{r}'; E) = \frac{1}{8}\int_0^{2\Pi} \mathrm{d}\phi\, U(\xi, \eta; Z)\, \mathrm{e}^{2\mathrm{i}q(\phi-\phi')} \tag{21}$$

is the energy representation of the Green function for a hydrogen-like atom in both the Dirac monopole field and the Aharonov–Bohm field with the presence of the scalar potential $-q^2/2r^2$.

3. The Green function for a hydrogen-like atom in both the Dirac monopole field and the Aharonov–Bohm field

A simple method of constructing the Green function $K_q(\boldsymbol{r}, \boldsymbol{r}'; E)$ results from the found connection (21). Starting from the well known expression for the Green function of a harmonic oscillator we write down the energy representation of the Green function for an isotropic harmonic oscillator in two-dimensional complex space

$$U(\xi, \eta; Z) = \frac{2\omega}{\pi^2}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}(2Z/\omega)t}(\sin t)^{-2} \times \exp\left\{\frac{\mathrm{i}\omega}{\sin t}[(\xi_s^*\xi_s + \eta_s^*\eta_s)\cos t - (\xi_s^*\eta_s + \eta_s^*\xi_s)]\right\}. \tag{22}$$

Changing the variables (15) and choosing as extra variables

$$\phi = \nu \arg(\xi_1) + \mu \arg(\xi_2)$$
$$\phi' = \nu \arg(\xi_1) + \mu \arg(\xi_2) \qquad \nu + \mu = 1 \tag{23}$$

we arrive at the formula (we use the following spherical coordinates in the spaces $\boldsymbol{r}$ and $\boldsymbol{r}'$: r, ϑ, φ and r', ϑ', φ')

$$U(r, \vartheta, \varphi, \phi, r', \vartheta', \varphi'; Z) = -\frac{2\omega}{\pi^2}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}(2Z/\omega)t}(\sin t)^{-2}\exp\left\{\mathrm{i}\omega(r+r')\cot t - 2\frac{\mathrm{i}\omega}{\sin t}\sqrt{rr'}\left(\cos\frac{\vartheta}{2}\cos\frac{\vartheta'}{2}\cos(\phi-\phi'-\mu\varphi+\mu\varphi') + \sin\frac{\vartheta}{2}\sin\frac{\vartheta'}{2}\cos(\phi-\phi'+\nu\varphi-\nu\varphi')\right)\right\}. \tag{24}$$

After integration over the extra variables ϕ and ϕ', we obtain

$$
\begin{aligned}
K_q(\boldsymbol{r}, \boldsymbol{r}'; E) = -\frac{\omega}{2\pi} \mathrm{e}^{-2\mathrm{i}q\mu(\varphi-\varphi')} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}(2Z/\omega)t} (\sin t)^{-2}\, \mathrm{e}^{\mathrm{i}\omega(r+r')\cot t} \\
\times \exp\left\{ 2\mathrm{i}q \tan^{-1} \frac{xy'-yx'}{rr'+\boldsymbol{r}\cdot\boldsymbol{r}'+r'x_3+rx_3'} \right\} \\
\times J_{2q}\left(\frac{\omega}{\sin t} (2(rr'+\boldsymbol{r}\cdot\boldsymbol{r}')^{1/2}) \right)
\end{aligned}
\tag{25}
$$

where $J_{2q}(x)$ is the Bessel function. This is a representation of the Green function for a hydrogen-like atom in the Dirac monopole field plus the Aharonov–Bohm field with the presence of the scalar potential $-q^2/2r^2$. In the limit (when (i) $q \to 0$, (ii) $q \neq 0$, $\mu \to 0$) the function (25) coincides with the well known results ((i) the Coulomb Green function (Le Van Hoang *et al* 1989), (ii) the Green function for a hydrogen-like atom in the Dirac monopole field (Le Van Hoang 1992)).

In conclusion it should be noted that the connection (21) can be established for the class of potentials created by a system of solenoids.

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